Optics for Energy
Week 7. Thursday

Symmetry (contd.) & etedue in phase space
Circular symmetry

$h = \text{skew invariant or skewness is conserved in a system with circular symmetry}$.

Plane $\nu$ is parallel to $\hat{\mathbf{e}}_2 \hat{\mathbf{e}}_3 \quad \Rightarrow \quad \phi$ is a constant.

So the component of momentum along $\hat{\mathbf{e}}_2$ doesn't change upon refraction.

$$\Rightarrow \quad p_\phi = n_1 c_1 \sin \beta_1 = n_2 c_2 \sin \beta_2$$

But $p_\phi$ is not constant upon propagation. $h$ is!

Projecting the ray onto plane $\nu$, we get

$$\sqrt{n_1^2 - p_\phi^2} \sin \beta_1 = \sqrt{n_2^2 - p_\phi^2} \sin \beta_2$$

$$h = n_1 c_1 \phi$$

$$n_1 c_1 \phi = p_\phi$$

$$\Rightarrow \quad p_\phi = h / \phi$$

$$\Rightarrow \quad \sqrt{n_1^2 - (h / \phi)^2} \sin \beta_1 = \sqrt{n_2^2 - (h / \phi)^2} \sin \beta_2$$
\[ \rho = | \overrightarrow{r} | \sin \beta = | \hat{e}_3 \times \overrightarrow{r} | \]

\[ \overrightarrow{r} = \overrightarrow{0} \]

Then, \[ h = | \hat{e}_3 \times \overrightarrow{r} | \frac{| \overrightarrow{p} |}{m} \cos \phi \]

\[ = \rho \cdot (\hat{e}_3 \times \overrightarrow{r}) \]

since \( \hat{e}_3 \times \overrightarrow{r} \) is \( \parallel \) to \( \hat{e}_0 \) (since \( h = \rho / m \cos \phi \))
\[ \vec{p}_p = \text{projection of } \vec{p} \text{ onto } x_1 x_2 \text{ plane (II to } \hat{e}_0 \cdot \hat{e}_p \text{ plane).} \]

\[ |\vec{p}_p| = n \sin \gamma \]

\[ \vec{r}_p = \text{projection of } \vec{r} \text{ onto } x_1 x_2 \text{ plane} \]

\[ |\vec{r}_p| = \rho \]

Then \[ h = n \rho \cos \phi = |\vec{p}_p| |\vec{r}_p| \sin \alpha \quad (\phi + \alpha = \pi/2) \]

\[ = |\vec{r}_p \times \vec{p}_p| \]
plane $\nu$ is parallel to $\hat{e}_z \hat{e}_r$ $\rightarrow \Theta$ is a constant.

So the component of momentum along $\hat{e}_z$ doesn't change upon refraction.

$$n_2 \cos \theta \alpha_{n_2} = n_2 \cos \alpha_{n_2}$$

But $P_\Theta$ is not constant upon propagation. $h$ is!

Projecting the ray onto plane $\nu$, we get

$$\frac{\sqrt{n_1^2 - P_\Theta^2}}{\sin \beta_1} = \frac{\sqrt{n_2^2 - P_\Theta^2}}{\sin \beta_2}$$

$$h = n_2 \cos \theta$$

$$n_2 \alpha_\phi = P_\Theta$$

$$\Rightarrow \quad P_\Theta = \frac{h}{n_2}$$

$$\Rightarrow \quad \frac{\sqrt{n_1^2 - (\frac{h}{n_2})^2}}{\sin \beta_1} = \frac{\sqrt{n_2^2 - (\frac{h}{n_2})^2}}{\sin \beta_2}$$
Étendue in phase space

Conservation of étendue

\[ dx_1 dx_2 dx_3 dx_4 dp_1 dp_2 \]

The quantity, \( dx_1 dx_2 dx_3 dx_4 dp_1 dp_2 \), is conserved as light travels through an optical system.

Example of free-space propagation

The quantity, \( dV \), \( dx_1 dx_2 dp_1 dp_2 \), is conserved as light travels through an optical system.

Example of refraction

The quantity, \( dV \), \( dx_1 dx_2 dp_1 dp_2 \), is conserved as light travels through an optical system.

Definition

The étendue of this bundle of rays crossing \( dS \) is defined as:

- \( d^2G := n dS \cos \theta \, d\theta \) in 2D space
- \( d^2G := n^2 dS \cos \theta \, d\theta d\Omega \) in 3D space
Conservation of etendue

\[ dx_1 dx_2 \, dp_1 dp_2 = dx_1^* dx_2^* dp_1^* dp_2^* \]

The quantity, \( dU = dx_1 dx_2 dp_1 dp_2 \) is conserved as light travels through an optical system.
The quantity, $dU = d\Omega_1 d\Omega_2 dp_1 dp_2$, is conserved as light travels through an optical system.
The quantity, $dU = dx_1 dx_2 dp_1 dp_2$ is conserved as light travels through an optical system.
Another way to look at etendue is

\((x_1, x_2, p_1, p_2)\) defines a point in phase space.

This is a point and a direction \(\Rightarrow\) Ray

A volume of points in phase space \(\Rightarrow\) Bundle of rays

The volume in phase space occupied by a bundle of rays is constant (or increasing) as it passes through an optical system.
The etendue of this bundle of rays crossing $dS$ is defined as:

$$d^2G := ndS \cos \theta d\theta$$  in 2D space

$$d^2G := n^2 dS \cos \theta d\Omega$$  in 3D space
Solid Angle Review

The solid angle subtended by an arbitrary surface to a point is given by:

\[ \Omega = \iiint_{S} \frac{\hat{r} \cdot \hat{n} \, dS}{r^3} = \iiint_{S} \sin \theta \, d\theta \, d\varphi. \]
Conservation of Etendue

Example of free-space propagation

Light source = \Sigma  
Refractive index of medium = n  
Receiver = S

The etendue of light crossing dΩ towards dS is:
\[ d^2G_{\Sigma} = n^2 dΩ \cos \theta_\Sigma d\Omega_{\Sigma} = n^2 dΩ \cos \theta_\Sigma \frac{ds \cos \theta_S}{ds} \]
where dΩ_\Sigma is the solid angle subtended by the area dS at area dΩ.

The etendue for the whole system is:
\[ G = \int_{S} \int_{\Sigma} d^2G \]

Etendue as a volume in phase space

Note a light ray is defined by:
1. Position. \((x, y, z)\)
2. Direction. \((\cos \alpha_X, \cos \alpha_Y, \cos \alpha_Z)\)
3. Refractive index, n.

The optical momentum at that point is defined as
\[ p = n(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \]

Etendue as a volume in phase space

In spherical coordinates,
\[ dp dq = \frac{\partial(p, q)}{\partial(\theta, \varphi)} d\theta d\varphi = \left( \frac{\partial p}{\partial \theta} \frac{\partial q}{\partial \varphi} - \frac{\partial p}{\partial \varphi} \frac{\partial q}{\partial \theta} \right) d\theta d\varphi \]
\[ = n^2 \cos \theta \sin \theta d\theta d\varphi = n^2 \cos \theta d\Omega \]

Therefore, etendue for an area dS = dx dy on the xy plane in medium of index, n is
\[ d^2G = n^2 dS \cos \theta d\Omega = dx \ dy \ dp \ dq \]
This is an infinitesimal volume in phase space \( x, y, p, q \).
Example of free-space propagation

Light source = $\Sigma$  
Refractive index of medium = $n$

Receiver = $S$  

The etendue of light crossing $d\Sigma$ towards $dS$ is:

$$d^2 G_{\Sigma} = n^2 d\Sigma \cos \theta_{\Sigma} d\Omega_{\Sigma} = n^2 d\Sigma \cos \theta_{\Sigma} \frac{dS \cos \theta_S}{d^2}$$
where $d\Omega_{\Sigma}$ is the solid angle subtended by the area $dS$ at area $d\Sigma$

The etendue of light crossing $dS$ coming from $d\Sigma$

$$d^2 G_{S} = n^2 dS \cos \theta_{S} d\Omega_{S} = n^2 dS \cos \theta_{S} \frac{d\Sigma \cos \theta_{\Sigma}}{d^2}$$
where $d\Omega_{S}$ is the solid angle subtended by the area $d\Sigma$ at area $dS$.

$$d^2 G_{\Sigma} = d^2 G_{S}$$  
The etendue is conserved!

The etendue for the whole system is:

$$G = \int_{\Sigma} \int_{S} d^2 G$$
Example of refraction

incident light

Snell's Law -> \( n_\Sigma \sin \theta_\Sigma = n_S \sin \theta_S \)

\( n_\Sigma \cos \theta_\Sigma d\theta_\Sigma = n_S \cos \theta_S d\theta_S \)

\( n_\Sigma^2 \cos \theta_\Sigma \left( \sin \theta_\Sigma d\theta_\Sigma d\varphi \right) = n_S^2 \cos \theta_S \left( \sin \theta_S d\theta_S d\varphi \right) \)

Note this term doesn't change upon refraction.

\( n_\Sigma^2 \cos \theta_\Sigma d\Omega_\Sigma = n_S^2 \cos \theta_S d\Omega_S \)

\( n_\Sigma^2 dS \cos \theta_\Sigma d\Omega_\Sigma = n_S^2 dS \cos \theta_S d\Omega_S \leftrightarrow d^2 G_\Sigma = d^2 G_S \)

Etendue is conserved upon refraction.

This is also easily proved for reflection in the same manner.
Etendue as a volume in phase space

Note a light ray is defined by:
1. Position. \((x, y, z)\)
2. Direction. \((\cos \alpha_x, \cos \alpha_y, \cos \alpha_z)\)
3. Refractive index, \(n\).

The optical momentum at that point is defined as

\[ p = n(\cos \alpha_x, \cos \alpha_y, \cos \alpha_z) = (p, q, r) \]
Etendue as a volume in phase space

In spherical coordinates,

\[ \mathbf{p} = n \left( \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \right) \]

\[ dp \, dq = \frac{\partial (p, q)}{\partial (\theta, \varphi)} \, d\theta \, d\varphi = \left( \frac{\partial p}{\partial \theta} \frac{\partial q}{\partial \varphi} - \frac{\partial p}{\partial \varphi} \frac{\partial q}{\partial \theta} \right) \, d\theta \, d\varphi \]

\[ = n^2 \cos \theta \sin \theta \, d\theta \, d\varphi = n^2 \cos \theta \, d\Omega \]

Therefore, etendue for an area \( dS = dx \, dy \) on the xy plane in medium of index, \( n \) is

\[ d^2 G = n^2 \, dS \, \cos \theta \, d\Omega = dx \, dy \, dp \, dq \]

This is an infinitesimal volume in phase space \( x, y, p, q \).
Example Problem.
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<tr>
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<td>Rays &amp; Wavefronts [Literature reviews due]</td>
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<td>Light tools optical design software tutorial (Guest lecture: Dr. Mohit Diwekar)</td>
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<td>Radiometry, Photometry, Radiation heat transfer</td>
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<td>Fundamental concepts of Non-imaging optics [Project plan section due]</td>
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<td>Fundamental concepts of Non-imaging optics</td>
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<td>Solar Simulators [National Clean Energy Business Plan Application due]</td>
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