Optics for Energy
Week 7 Tuesday
Symmetry
\[ P_2 = P_1 - \hat{n} \left( (P_1 \cdot \hat{n}) + \sqrt{(P_1 \cdot \hat{n})^2 - n_1^2 + n_2^2} \right) \]

\[ P_2 = P_1 - \hat{n}_2 (P_1 \cdot \hat{n}) \]
Now consider a generalized co-ordinate system $i_1, i_2, i_3$.

\[ n^2(1 - \cos^2 \alpha_3)\cos^2 \beta_2 = n^2 \cos^2 \alpha_2 \]

\[ (n^2 - P_3^2)\cos^2 \beta_2 = P_2^2 \]
Orient the axes such that the normal at the point of interest point in the $i_1$ direction.

$$
(n_1^2 - P_3^2(n_1)) \cos^2 \beta_{2n_1} = P_2^2(n_1) \\
(n_2^2 - P_3^2(n_2)) \cos^2 \beta_{2n_2} = P_2^2(n_2)
$$

$$
P_2(n_1) = P_2(n_2) \quad P_3(n_1) = P_3(n_2) \\
\Rightarrow \quad (n_1^2 - P_3^2) \cos^2 \beta_{2n_1} = (n_2^2 - P_3^2) \cos^2 \beta_{2n_2}
$$

$$(n_k^*)^2 = (n_k^2 - P_3^2) \quad k = 1, 2$$

$$
n_1^* \sin \beta_1 = n_2^* \sin \beta_2$$

The projection of the ray onto the $i_1 i_2$ plane also obeys Snell's law if $n_k$ is replaced by $\sqrt{n_k^2 - P_3^2}$.
Consider a more general situation, where a curved surface separates the regions of the two refractive indices. Axis i3 is tangential to the surface. The normal lies in the i1i2 plane but not necessarily parallel to either one.

In this case, P3 is conserved upon refraction.

\[ P_3 = n_1 \cos \alpha_{3n1} = n_2 \cos \alpha_{3n2} \]
This situation represents linear symmetry. We can study this as a 2D system where the projections of light rays onto the i1-i2 plane behaves as a 2D system with refractive index $n^* = \sqrt{n^2 - n_3^2}$.

In this case, $P_3$ is conserved upon refraction.

$$P_3 = n_1 \cos \alpha_{3n_1} = n_2 \cos \alpha_{3n_2}$$
Consider now an optical system aligned along $x_3$, where $n$ does not change along $x_3$. $n = n(x_1, x_2)$.

$$\mathcal{P} = P_1^2 + P_2^2 + P_3^2 - n^2 = 0$$

Hamiltonian

$$\frac{\partial \mathcal{P}}{\partial x_3} = \frac{\partial n}{\partial x_3} = 0$$

From the canonical equations, we have

$$-\frac{\partial \mathcal{P}}{\partial x_3} = \frac{dP_3}{d\sigma} = 0$$

or $P_3$ is a constant.

Let $P_3 = C$.

This implies that

$$x_3 = 2P_3 \sigma + C$$
Rewriting the canonical equations.

\[
\frac{dx_i}{d\sigma} = \frac{\partial \hat{P}}{\partial \hat{p}_i} \quad \frac{dp_i}{d\sigma} = -\frac{\partial \hat{P}}{\partial x_i}
\]

\[
\frac{dx_2}{d\sigma} = \frac{\partial \hat{P}}{\partial \hat{p}_2} \quad \frac{dp_2}{d\sigma} = -\frac{\partial \hat{P}}{\partial x_2}
\]

The analysis of a 3D system with linear symmetry can be reduced to that of a 2D system.

This is a particular case where the Hamiltonian doesn’t depend on one coordinate. This coordinate is then referred to as cyclic. And the system can be described by one less dimension.

\[
P = \hat{p}_1^2 + \hat{p}_2^2 - (\hat{x}^*)^2 = 0
\]

where \((\hat{x}^*)^2 = \hat{v}^2(\hat{x}_1, \hat{x}_1) - \hat{p}_3^2\)

\L_{\hat{p}_3} = \text{constant}.
Note the Euler equation can be simplified as well.

\[ \frac{\partial L}{\partial x_k} = - \frac{\partial L}{\partial x_k} = 0 \]

when \( x_k \) is cyclic.

The Euler equation becomes ->

\[ \frac{d}{d\sigma} \left( \frac{\partial L}{\partial x_k} \right) = 0 \]

\[ \Rightarrow \frac{\partial L}{\partial x_k} = \text{constant.} \quad = P_k \]
Circular symmetry & Skew invariance

Many optical systems exhibit circular or axial symmetry, i.e., they are symmetric around an axis of rotation, say x3.
In this case, \( n \) = normal to surface of refraction or reflection has no component along \( \hat{e}_0 \).

Also, the component of \( P \) along \( \hat{e}_0 \) will be conserved in reflection and refraction.

\[
\Rightarrow \quad P_0 = n \cos \phi \quad \text{where} \quad \phi \quad \text{is the angle between light ray and} \quad \hat{e}_0
\]

\[
\Rightarrow \quad n_1 \cos \phi_1 = n_2 \cos \phi_2
\]
But note that $\rho_a$ can change as the ray propagates (no reflection or refraction) because $\phi$ changes.

$\rho = \text{projection of } r \text{ onto } x_1-x_2 \text{ plane.}$

$\rho \sin \alpha = M = \text{constant}

\text{perpendicular to optical axis from ray.}$
\[ |\overrightarrow{OC}| = \cos \phi = |\overrightarrow{OB}| \sin \alpha = \sin \gamma \sin \alpha \]

\[ h = n \rho \sin \alpha \sin \gamma = n M \sin \gamma = n \rho \cos \phi \]

\[ P_0 = n \cos \phi = \frac{h}{\rho} = b \rho h \quad , \quad b = \frac{1}{\rho} \]

\[ \Rightarrow \quad n_1 \rho \cos \phi_1 = n_2 \rho \cos \phi_2 \]

Conserved

h is also conserved in refractions & reflections.
h = skew invariant or skewness
Generalized laws of reflection & refraction

The interface between the two media is artificially structured in order to introduce an abrupt phase shift in the light path, which is a function of position along the interface.

Applying Fermat's principle, minimize $\int_A^B n(\vec{r}) \, dr$.

Fermat's principle can also be applied as the principle of stationary phase, i.e., the derivative of the phase accumulated along the light path will be zero with respect to infinitesimal variations of the path.
If abrupt phase shifts are introduced at the interface, then the total phase will be stationary for the actual path that the light ray takes.

Total phase of ray = \( \Phi(\vec{r}_s) + \int_A \vec{k} \cdot d\vec{r} \)

where \( \Phi(\vec{r}_s) \) is the abrupt phase change at the interface
\( \vec{r}_s \) is the coordinate along the interface.
\( \vec{k} \) is the wave-vector of the ray.

If the blue and red rays are infinitesimally close to the actual path that light takes, then the phase difference between them must be zero.
If the blue and red rays are infinitesimally close to the actual path that light takes, then the phase difference between them must be zero.

\[
\left[ k_o n_i \sin(\theta_i) dx + (\Phi + d\Phi) \right] - \left[ k_o n_t \sin(\theta_t) dx + \Phi \right] = 0
\]

\[ k_o = \frac{2\pi}{\lambda_c} \]

This gives rise to the Generalized law of refraction.

\[
\sin(\theta_t)n_t - \sin(\theta_i)n_i = \frac{\lambda_o}{2\pi} \frac{d\Phi}{dx}
\]
One interesting consequence of this is that the two angles of incidence $\pm \theta_i$ lead to different angles of refraction.

Also, there are two possible critical angles, when $n_t < n_i$:

$$\theta_c = \arcsin \left( \pm \frac{n_t}{n_i} - \frac{\lambda_o}{2\pi n_i} \frac{d\Phi}{dx} \right)$$
Experimentally realized with plasmonic (resonant) antennae
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Project Deliverables (~3 pages each)

Mini-report 1: Literature Review; Discuss pros, cons, comparisons, overview of existing technologies. Due 09/13/2012


Mini-report 5: Business Plan, Commercialization strategy. Due 12/06/2012

Deadlines are important! 10% of points docked for each day you are late.