Optics for Energy
Lecture 8. Tuesday
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Lagrangian Optics
Fermat's Principle
Light travels between 2 points so as to minimize the travel time.

optical path length (A to B)
\[ L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (d-x)^2} \]

Minimize distance (c = constant)
\[ \frac{dL}{dx} = \frac{2x}{2 \sqrt{a^2 + x^2}} + \frac{1}{2} \frac{2(d-x)(-1)}{2 \sqrt{b^2 + (d-x)^2}} = 0 \]

This reduces to
\[ \frac{x}{\sqrt{a^2 + x^2}} = \frac{(d-x)}{\sqrt{b^2 + (d-x)^2}} \]
which is \( \sin \theta_i = \sin \theta_r \)

This shows that:
\[ \theta_i = \theta_r \]
Law of Reflection
But minimum distance alone doesn’t explain refraction

That is the minimum time principle.

\[ t = \frac{\sqrt{a^2 + x^2}}{v} + \frac{\sqrt{b^2 + (d - x)^2}}{v} \]

\[ \frac{dt}{dx} = \frac{x}{v\sqrt{a^2 + x^2}} - \frac{(d-x)}{v\sqrt{b^2 + (d-x)^2}} \]

\[ 0 = \frac{\sin \theta_1}{v} - \frac{\sin \theta_2}{v} \]

**Snell's Law**
\[ \frac{n_1}{n_2} = \frac{\sin \theta_2}{\sin \theta_1} \]
Optical Path Length = n X physical path length

Minimum time = minimum optical path length (explains both reflection & refraction).

\[ P_1 P_2 = S_1 \quad \text{actual path of light ray} \]

\[ P_1 P_2 = S_2 \quad \text{possible path of light ray} \]

\[ S_2 = P_1 P_2 = P_1 A + ds_1 + AB - ds_2 = S_1 + ds_1 - ds_2 \]

\[ ds_1 = ds_2 \]

\[ S_2 = S_1 \] for infinitesimal \( dx \)

or \( ds = 0 \)
In general, $n$ can vary with space (e.g., transformation optics, metamaterials, etc.)

\[ OPL = \int_a^b n(s) ds \]

\[ OPL = nd \]
A curve in 3D is defined by 1 parameter & 3 components

An arbitrary trajectory of light is defined by a curve in 3D space: \( S \)

\[
S(\sigma) = S(x_1(\sigma), x_2(\sigma), x_3(\sigma))
\]

\[
P_1 = \{ x_1(\sigma_1), x_2(\sigma_1), x_3(\sigma_1) \}
\]

\[
P_2 = \{ x_1(\sigma_2), x_2(\sigma_2), x_3(\sigma_2) \}
\]

The infinitesimal curve length \( ds \) is given by

\[
ds = \sqrt{dx_1^2 + dx_2^2 + dx_3^2}
\]

\[
ds = \sqrt{\left(\frac{dx_1}{d\sigma}\right)^2 + \left(\frac{dx_2}{d\sigma}\right)^2 + \left(\frac{dx_3}{d\sigma}\right)^2} \ d\sigma
\]

Then, the optical path length \( S \) is

\[
S = \int_{\sigma_1}^{\sigma_2} n \ ds = \int_{\sigma_1}^{\sigma_2} \left\{ n \sqrt{\left(\frac{dx_1}{d\sigma}\right)^2 + \left(\frac{dx_2}{d\sigma}\right)^2 + \left(\frac{dx_3}{d\sigma}\right)^2} \right\} \ d\sigma
\]

Now for \( S \) to be a light ray, the optical path length must be minimized or \( \delta S = 0 \). The optical path length is said to be stationary along a light ray.

\[
\delta S = \delta \int_{\sigma_1}^{\sigma_2} n \ ds = \delta \int_{\sigma_1}^{\sigma_2} \left\{ n \sqrt{\left(\frac{\partial x_1}{\partial \sigma}\right)^2 + \left(\frac{\partial x_2}{\partial \sigma}\right)^2 + \left(\frac{\partial x_3}{\partial \sigma}\right)^2} \right\} \ d\sigma
\]
Define the Lagrangian

\[ L = n \sqrt{\left( \frac{\partial x_1}{\partial \sigma} \right)^2 + \left( \frac{\partial x_2}{\partial \sigma} \right)^2 + \left( \frac{\partial x_3}{\partial \sigma} \right)^2} \]

Then, the condition for \( S \) to define a light ray is:

\[ \delta S = \delta \int_{\sigma_1}^{\sigma_2} n \, ds = \delta \int_{\sigma_1}^{\sigma_2} L(x_1, x_2, x_3, x'_1, x'_2, x'_3) \, d\sigma \]

where \( x'_i = \frac{\partial x_i}{\partial \sigma}, i = 1,2,3. \)

Let's take the example of refraction

Let \( S_1 = P_1 A P_2 \) & \( S_2 = P_1 B P_2 \)

Then, \( \delta S = P_1 A P_2 - P_1 B P_2 \)

\[ \delta S = AB \]

\[ \delta S = n_1 t \, \delta S - n_2 \, \delta S = (n_1 t - n_2 \, \hat{n}) \, \delta S = 0 \quad \text{for a ray.} \]

But, we know that \( k \, \hat{n} \cdot \delta S = 0 \)
Therefore, \((n_1 \hat{t} - n_2 \hat{r}) = k \hat{n}\)

\[\Rightarrow\] All 3 vectors are co-planar.

\[(n_1 \hat{t} - n_2 \hat{r}) \times \hat{n} = k \hat{n} \times \hat{n} = 0\]

\[n_1 \hat{t} \times \hat{n} = n_2 \hat{r} \times \hat{n}\]

\[n_1 \sin \alpha_1 = n_2 \sin \alpha_2\]
Lagrangian Formulation

2D optical path from $P_1$ to $P_2$: $S = \int_{P_1}^{P_2} L(x_1, x_2, x'_1, x'_2) d\sigma$

$x'_i = \frac{\partial x_i}{\partial \sigma}, i = 1, 2.$

Consider 2 infinitesimally close curves

$C_2 = C_1 + \eta \, S\chi$

$\eta = (\eta_1(\sigma), \eta_2(\sigma))$

$S\chi = \text{infinitesimal distance}$

$Sx_1 = \eta_1(\sigma) \, S\chi$

$Sx_2 = \eta_2(\sigma) \, S\chi$

Since the curves pass through the same points $P$ and $P$.

$\eta_1(\sigma_1) = \eta_2(\sigma_1) = 0$

$\eta_1(\sigma_2) = \eta_2(\sigma_2) = 0$
Now, let's consider the optical path length of $C_2$.

\[ S_2 = \int_{\sigma_1}^{\sigma_2} \left( I + \frac{\partial I}{\partial x_2} \eta_2 \delta \alpha + \frac{\partial I}{\partial x_2} \frac{\partial \alpha}{\partial x_2} \eta_2 \delta \alpha + \frac{\partial I}{\partial x_2} \frac{\partial \alpha}{\partial x_2} \frac{\partial \alpha}{\partial x_2} \eta_2 \delta \alpha \right) \, ds \]

where

\[ \eta_2 = \frac{\partial \eta_2}{\partial \sigma}, \quad f' h' = 1, 2. \]

And the optical path length difference between the two curves is:

\[ S_2 - S_1 = \int_{\sigma_1}^{\sigma_2} \left( \frac{\partial I}{\partial x_2} \frac{\partial \alpha}{\partial x_2} \eta_2 \delta \alpha + \frac{\partial I}{\partial x_2} \frac{\partial \alpha}{\partial x_2} \frac{\partial \alpha}{\partial x_2} \eta_2 \delta \alpha \right) \, ds \]
Now, we can further simplify these terms.

\[
\int_{\sigma_1}^{\sigma_2} \left( \frac{\partial \frac{\partial X}{\partial x_1}}{\partial x_1} h_k \right) d\sigma = \int_{\sigma_1}^{\sigma_2} \left[ \frac{d}{d\sigma} \left( \frac{\partial X}{\partial x_1} h_k \right) - \frac{d}{d\sigma} \left( \frac{d}{d\sigma} \left( \frac{\partial X}{\partial x_1} \right) h_k \right) \right] d\sigma
\]

\[h = 1, 2\]

\[
\int_{\sigma_1}^{\sigma_2} \frac{d}{d\sigma} \left( \frac{\partial X}{\partial x_1} h_k \right) d\sigma = \left. \int_{\sigma_1}^{\sigma_2} \frac{d}{d\sigma} \left( \frac{\partial X}{\partial x_1} h_k \right) \right|_{\sigma_1}^{\sigma_2}
\]

\[
= \frac{\partial X}{\partial x_1}(\sigma_2) h_k(\sigma_2) - \frac{\partial X}{\partial x_1}(\sigma_1) h_k(\sigma_1) = 0
\]

since \( h_k(\sigma_2) = h_k(\sigma_1) = 0 \)

\[
\Rightarrow \int_{\sigma_1}^{\sigma_2} \frac{\partial X}{\partial x_1} h_k^1 d\sigma = - \int_{\sigma_1}^{\sigma_2} \left[ \frac{d}{d\sigma} \left( \frac{d}{d\sigma} \left( \frac{\partial X}{\partial x_1} \right) h_k \right) \right] d\sigma
\]

Then, \( SS = \int_{\sigma_1}^{\sigma_2} \left\{ \left( \frac{\partial X}{\partial x_1} - \frac{d}{d\sigma} \left( \frac{\partial X}{\partial x_1} \right) \right) h_1 + \left( \frac{\partial X}{\partial x_2} - \frac{d}{d\sigma} \left( \frac{\partial X}{\partial x_2} \right) \right) h_2 \right\} d\sigma \)
Now $\delta S = 0$ for any arbitrary function $\eta$

$$\delta S = \int_{\sigma_i}^{\sigma_f} \left\{ \left( \frac{d}{d\sigma} \left( \frac{\partial L}{\partial \dot{x}_1} \right) \right) \eta \right\} \, d\sigma$$

Each of the two terms under the integral must be 0.

$$\frac{\partial L}{\partial x_1} = \frac{d}{d\sigma} \left( \frac{\partial L}{\partial \dot{x}_1} \right)$$

Euler equations for the curve

$$\frac{\partial L}{\partial x_2} = \frac{d}{d\sigma} \left( \frac{\partial L}{\partial \dot{x}_2} \right)$$

This approach to computing the curve is called the Lagrangian Formulation.