The Hamiltonian Formulation

Let us define 2 new variables called momenta:

\[ P_1 \equiv \frac{\partial \mathcal{L}}{\partial x_1'} \quad P_2 \equiv \frac{\partial \mathcal{L}}{\partial x_2'} \]

The Euler equations can now be written as:

\[ \frac{dP_1}{d\sigma} = \frac{\partial \mathcal{L}}{\partial x_1} \quad \frac{dP_2}{d\sigma} = \frac{\partial \mathcal{L}}{\partial x_2} \]

Let us define a new function, \( H \):

\[ H \equiv x_1' P_1 + x_2' P_2 - \mathcal{L} \]

Note that

\[ x_1' = \int \frac{1}{P_1} \partial \mathcal{L} \]

\[ \mathcal{L} = \mathcal{L}(x_1, x_2, x_1', x_2', \sigma) \]

\[ \Rightarrow x_1' = x_1'(x_1, x_2, P_1, P_2, \sigma) \quad \& \quad x_2' = x_2'(x_1, x_2, P_1, P_2, \sigma) \]

\[ H = H(x_1, x_2, P_1, P_2, \sigma) \]
The differential of $H$

\[ dH = \sum_{k=1,2} \frac{\partial H}{\partial x_k} dx_k + \frac{\partial H}{\partial P_k} dP_k + \frac{\partial H}{\partial \sigma} d\sigma \]

From the definition of $H$: \[ H \equiv x'_1 P_1 + x'_2 P_2 - \mathcal{L} \]

\[ dH = \sum_{k=1,2} dP_k x'_k + dx'_k P_k - \frac{\partial \mathcal{L}}{\partial x'_k} dx'_k - \frac{\partial \mathcal{L}}{\partial x_k} dx_k - \frac{\partial \mathcal{L}}{\partial \sigma} d\sigma \]

\[ dH = \sum_{k=1,2} dP_k x'_k - P_k dx'_k - \frac{\partial \mathcal{L}}{\partial \sigma} d\sigma \]
Comparing the 2 expressions of $dH$

$$dH = \sum_{k=1,2} \frac{\partial H}{\partial x_k} dx_k + \frac{\partial H}{\partial P_k} dP_k + \frac{\partial H}{\partial \sigma} d\sigma$$

$$dH = \sum_{k=1,2} dP_k x'_k - P'_k dx_k - \frac{\partial L}{\partial \sigma} d\sigma$$

Recall that $x'_k = \frac{\partial x_k}{\partial \sigma}$

Canonical Hamilton equations whose Hamiltonian is $H$. 

$k = 1,2$
Lagrangian vs Hamiltonian

In the Lagrangian formulation, the path of a light ray is calculated in so-called configuration space $(x, \dot{x})$. The path is defined by the Euler equations, two 2nd order partial-differential equations (PDEs).

In the Hamiltonian formulation, the path of a light ray is calculated in so-called phase space $(x_1, \dot{x}, p_1, p_2)$. The path is defined by the canonical Hamilton equations, four 1st order PDEs.